A Correspondence Between Distances and Embeddings for Manifolds: New Techniques for Applications of the Abstract Boundary

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Abstract

We present a one-to-one correspondence between equivalence classes of embeddings of a manifold (into a larger manifold of the same dimension) and equivalence classes of certain distances on the manifold. This correspondence allows us to use the Abstract Boundary to describe the structure of the 'edge' of our manifold without resorting to structures external to the manifold itself. This is particularly important in the study of singularities within General Relativity where singularities lie on this 'edge'. The ability to talk about the same objects, e.g., singularities, via different structures provides alternative routes for investigation which can be invaluable in the pursuit of physically motivated problems where certain types of information are unavailable or difficult to use.

1 Introduction

The study of singularities within General Relativity suffers from a unique problem in physics: there is no background metric in which the singularity exists. Yet our intuition wishes to describe these 'singularities' with a location and physical properties. There are also the additional problems of providing a co-ordinate independent definition of a singularity and a description of the full range of singular behaviour.

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There are a number of boundary constructions which attempt to provide both a definition of and a location for singularities of space-times (see [1] for a review of the most notable boundary constructions, [17] for a review of the field in general and [3] for proofs of the most important applications) for a review of the field in general). The three most common examples are the g-boundary [7], b-boundary [15] and the c-boundary [8][1]. Each of these uses some aspects of the metric structure to identify 'missing' points from the space-time and then prescribes a method to re-attach them. Because each of these constructions uses the metric structure in this way, they each suffer from a variety of flaws. The Abstract Boundary (or a-boundary) [16] avoids these flaws by only using topological information in its construction.

The Abstract Boundary avoids using the metric structure by using embeddings as a kind of reference for boundary structure. Specifically, it uses the set of all embeddings $\phi : \mathcal{M} \rightarrow \mathcal{M}_\phi$, where $\mathcal{M}$ is the manifold of our original space-time and $\mathcal{M}_\phi$ is a manifold of the same dimension as $\mathcal{M}$ (we shall refer to such an embedding as an envelopment), to construct a set of equivalence classes of boundary points of these embeddings. Each equivalence class corresponds to the representation of a 'missing' point in a co-ordinate chart. Hence a 'missing' point may look very different depending on the chosen chart. Remarkably, a definition and location of a singularity can be retrieved from this rather general set-up.

We note that there have been two important recent contributions in this area. The first is García-Parrado and Senovilla's isocausal boundary [4, 5, 6] which is, in part, inspired from the Abstract Boundary. The second is a number of recent developments of the c-boundary; we refer the reader to Sánchez' interesting paper [14]. In both cases their work was more directly concerned with causal structures.

The very nice thing about the a-boundary is that it avoids all the usual problems inherent in metrically constructed boundaries. Unfortunately, this comes at a cost. In particular, complete knowledge of the a-boundary is reliant on knowing all possible envelopments of $\mathcal{M}$. This reliance makes it almost impossible to construct the complete a-boundary for a general space-time. It should be emphasised, however, that this does not, in any way, hinder its utility in the investigation of problems related to singularities (e.g., [12]). In much the same way as one does not need to know all charts in the atlas of a manifold, so too one does not need the full Abstract Boundary to extract information about the 'edge' of space-times. This is more a matter of representation than of missing information.

This paper demonstrates that there is a one-to-one correspondence between the set of equivalence classes of envelopments and a set of equivalence classes of distances on the manifold $\mathcal{M}$. We give a short example of how this correspondence can be used to investigate the a-boundary structure of space-times. In a future paper the authors will demonstrate that this correspondence can be used to construct the complete a-boundary from this set of equivalence classes of distances. Thus the correspondence provides an alternative method

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[1]See the preprint, [10], for an up to date review of the c-boundary including work conducted at the same time as this paper.

for studying the a-boundary. While this does not solve the problem mentioned above, it does make it more readily accessible.

We also hope that this correspondence will be of interest to all mathematicians desiring to study the 'edges' of manifolds. The work below demonstrates that the a-boundary has a strong relationship to Cauchy structures, in the sense of [11], and thus also to the more normal boundaries, e.g., the Stone-Čech compactification, employed in topology.

Section 2 introduces the necessary background for the Abstract Boundary. Sections 3 and 4 present equivalence relations on the set of all envelopments of a manifold and a set of distances on a manifold, respectively, and discuss the relation of this work to Cauchy structures. The first relation describes when two envelopments provide the same information about the a-boundary. The second relation mirrors the ideas of the first, but on a set of distances rather than envelopments. Section 5 gives a one-to-one correspondence between the two sets of equivalence classes, thereby showing that what can be constructed using the first can also be constructed using the second. Section 6 presents a short example of how this correspondence can be used.

Our main result, contained in section 5, is that the set of equivalence classes of envelopments, relevant for the Abstract Boundary, is in one-to-one correspondence with a set of equivalence classes of distances. So, in effect, the main result states that, in order to study the Abstract Boundary, one can use either envelopments or a certain subset of distances. The ability to employ distances when using the Abstract Boundary to investigate problems should provide both greater flexibility and accessibility.

To construct this correspondence we will use three homeomorphisms, between the closures, $\overline{\phi(\mathcal{M})}$, $\overline{\psi(\mathcal{M})}$, of the images of $\mathcal{M}$ under equivalent envelopments, ϕ, ψ, between the Cauchy completion, $\mathcal{M}^d, \mathcal{M}^{d'}$, of $\mathcal{M}$ of equivalent distances, d, d' and between the closure, $\overline{\phi(\mathcal{M})}$, of the image of $\mathcal{M}$ under an envelopment ϕ and the Cauchy completion of $\mathcal{M}$ with respect to a distance, d_ϕ, that is related to the envelopment. The existence of these homeomorphisms is ensured by propositions 3.5, 4.8 and corollary 4.9. The propositions both show that certain functions have extensions into the completion of their domains. It is here that the theory of Cauchy spaces underlies our result as the functions we consider are not necessarily uniformly continuous and therefore their well known extension theorem does not apply. The needed generalisation of this extension theorem is expressed in the language of Cauchy spaces; see subsections 3.1 and 4.1.

2   Preliminary results and notation

We need a few results and definitions from previous papers about the Abstract Boundary; they are collected below for the convenience of the reader. We recommend that the reader refer to the cited papers for a detailed introduction to the subject.

Definition 2.1 (see [16])**.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be manifolds of the same dimension. If there exists $\phi : \mathcal{M} \to \mathcal{M}'$ a C^∞ embedding, then $\mathcal{M}$ is said to be enveloped by $\mathcal{M}'$, $\mathcal{M}'$ is the enveloping manifold and ϕ is an envelopment. Since both manifolds have the same dimension, $\phi(\mathcal{M})$ is open in $\mathcal{M}'$.*

Definition 2.2 (see [16]). *Let $\phi : \mathcal{M} \to \mathcal{M}_\phi$ be an envelopment. A non-empty subset B of $\partial(\phi(\mathcal{M}))$ is called a boundary set.*

Definition 2.3 (see [16]). *A boundary set $B \subset \partial(\phi(\mathcal{M}))$ is said to cover another boundary set $B' \subset \partial(\psi(\mathcal{M}))$ if and only if for every open neighbourhood U of B in $\mathcal{M}_\phi$ there exists an open neighbourhood V of B' in $\mathcal{M}_\psi$ so that*

$$\phi \circ \psi^{-1}(V \cap \psi(\mathcal{M})) \subset U.$$

We shall denote this partial order by $\rhd$, so that B covers B' is written $B \rhd B'$. If a boundary set B is such that it consists of a single point, i.e., $B = \{p\}$, then we shall simply write $p \rhd B'$ rather than the more cumbersome $\{p\} \rhd B'$.

Theorem 2.4 (see [16]). *A boundary set $B_\phi \subset \partial(\phi(\mathcal{M}))$ covers another boundary set $B_\varphi \subset \partial(\varphi(\mathcal{M}))$ if and only if for every sequence $\{x_i\} \subset \mathcal{M}$ so that $\{\varphi(x_i)\}$ has an accumulation point in B_φ, the sequence $\{\phi(x_i)\}$ has an accumulation point in B_ϕ.*

Definition 2.5 (see [16]). *Two boundary sets $B \subset \partial(\phi(\mathcal{M}))$ and $B' \subset \partial(\psi(\mathcal{M}))$ are equivalent if and only if $B \rhd B'$ and $B' \rhd B$. We shall denote equivalence by $B \equiv B'$, and, as before, if $B = \{p\}$ then we shall simply write $p \equiv B'$ rather than $\{p\} \equiv B'$.*

We shall denote the equivalence class of the boundary set B by $[B]$. That is, $[B] = \{B' \subset \partial(\varphi(\mathcal{M})) : B' \equiv B$, where φ is an envelopment of $\mathcal{M}\}$. As usual, if $B = \{p\}$, then we shall write $[p]$ rather than $[\{p\}]$.

Definition 2.6 (see [16]). *The Abstract Boundary (a-boundary), $\mathcal{B}(\mathcal{M})$, of a manifold $\mathcal{M}$ is the set of all equivalence classes of boundary sets that contain a singleton, $p \in \partial(\phi(\mathcal{M}))$. That is,*

$$\mathcal{B}(\mathcal{M}) = \{[p] : p \in \partial(\phi(\mathcal{M})), \text{ where } \phi \text{ is an envelopment of } \mathcal{M}\}.$$

Definition 2.7 (see [1, 18]). *Two boundary sets $B \subset \partial(\phi(\mathcal{M}))$ and $B' \subset \partial(\psi(\mathcal{M}))$ are said to be in contact if for all open neighbourhoods U of B and V of B' we have that*

$$\phi^{-1}(U \cap \phi(\mathcal{M})) \cap \psi^{-1}(V \cap \psi(\mathcal{M})) \neq \varnothing.$$

Lemma 2.8 (see [1, 18]). *Let $B \subset \partial(\phi(\mathcal{M}))$ and $B' \subset \partial(\psi(\mathcal{M}))$ be boundary sets. Then B and B' are in contact if and only if there exists a sequence $\{x_i\}$ in $\mathcal{M}$ so that $\{\phi(x_i)\}$ has a limit point in B and $\{\psi(x_i)\}$ has a limit point in B'.*

Definition 2.9 (see [1, 18]). *Let $\phi : \mathcal{M} \to \mathcal{M}_\phi$ be an envelopment, then the set $\sigma_\phi = \{[p] \in \mathcal{B}(\mathcal{M}) : p \in \partial(\phi(\mathcal{M}))\}$, is the partial cross section of ϕ.*

We remind the reader that if $\mathfrak{s}$ is a sequence in $\mathcal{M}$, we mean that $\mathfrak{s} \subset \mathcal{M}$ and that $\mathfrak{s}$ is countable. We will not worry about a specific ordering of $\mathfrak{s}^2$.

By $\mathfrak{s} \to A$ we mean that there exists $x \in A$, a not necessarily unique, accumulation point of $\mathfrak{s}$. Where $A = \{x\}$ we shall write $\mathfrak{s} \to x$. The reason for this non-standard

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²The reason for this will be made clear in definition 4.2.

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notation is that, as points and sets are treated equivalently with respect to the Abstract Boundary, we are often interested in showing that a sequence has at least one limit point in a particular set but not that the sequence only has limit points in that set.

We will sometimes say that a sequence has a limit point. By this we mean only that a limit point exists. Where we need to mention unique limit points we say that the sequence $\mathfrak{s}$ converges to x or that $\mathfrak{s}$ has the unique limit point x.

3   An equivalence on the set of envelopments

We wish to define an equivalence relation that tells us when two envelopments are equivalent from the point of view of the Abstract Boundary: that is, when they produce the same Abstract Boundary points. There is a natural way to express this equivalence.

Definition 3.1. *Let $\mathcal{M}$ be a manifold, let $\phi : \mathcal{M} \to \mathcal{M}_\phi$ and $\psi : \mathcal{M} \to \mathcal{M}_\psi$ be two envelopments of $\mathcal{M}$. The envelopments ϕ and ψ are equivalent, $\phi \simeq \psi$, if and only if $\sigma_\phi = \sigma_\psi$. That is, $\phi \simeq \psi$ if and only if for all $x \in \partial(\phi(\mathcal{M}))$ there exists $y \in \partial(\psi(\mathcal{M}))$ so that $[y] = [x]$ and likewise for all $y \in \partial(\psi(\mathcal{M}))$ there exists $x \in \partial(\phi(\mathcal{M}))$ so that $[x] = [y]$.*

Proposition 3.2. *The equivalence relation $\simeq$ is well defined on the set of all envelopments.*

Proof. This follows from the fact that $=$ is a well defined equivalence relation on $\mathcal{B}(\mathcal{M})$. $\square$

Looking ahead, however, we shall be working with distances on $\mathcal{M}$ and will need a different, yet equivalent, definition that is easier to use in this setting. With this in mind we provide the following definition and result.

Definition 3.3. *Let $\Sigma_0(\mathcal{M}) = \{\mathfrak{s} : \mathfrak{s}$ is a sequence in $\mathcal{M}$ with no limit points in $\mathcal{M}\}$. Where the context is clear, we will drop the $\mathcal{M}$ and simply write Σ_0. Let $\phi : \mathcal{M} \to \mathcal{N}$ be an envelopment and $A \subset \partial(\phi(\mathcal{M}))$. Define $\Sigma(\phi, A)$ to be the set $\{\mathfrak{s} \in \Sigma_0(\mathcal{M}) : \phi(\mathfrak{s}) \to A\}$. We will often be interested in the case when $A = \{a\}$, where $a \in \partial(\phi(\mathcal{M}))$, and will write $\Sigma(\phi, a)$ rather than $\Sigma(\phi, \{a\})$.*

The following lemma will allow us to give definition 3.1 in terms of sequences in $\mathcal{M}$.

Lemma 3.4. *Let $\mathcal{M}$ be a manifold, let $\phi : \mathcal{M} \to \mathcal{M}_\phi$ and $\psi : \mathcal{M} \to \mathcal{M}_\psi$ be two envelopments of $\mathcal{M}$. Then $\sigma_\phi \subset \sigma_\psi$ if and only if for all $x \in \partial(\phi(\mathcal{M}))$ there exists $y \in \partial(\psi(\mathcal{M}))$ so that $\Sigma(\phi, x) = \Sigma(\psi, y)$[3].*

Proof. Suppose that $\sigma_\phi \subset \sigma_\psi$. Let $x \in \partial(\phi(\mathcal{M}))$, then $[x] \in \sigma_\phi$, so there exists $y \in \partial(\psi(\mathcal{M}))$ so that $[x] = [y]$; that is, $x \equiv y$. Now, let $\mathfrak{s} \in \Sigma(\phi, x)$, then $\phi(\mathfrak{s}) \to x$, but $y \triangleright x$, so $\psi(\mathfrak{s}) \to y$ and $\mathfrak{s} \in \Sigma(\psi, y)$. Therefore $\Sigma(\phi, x) \subset \Sigma(\psi, y)$. Since $x \triangleright y$ we can use the same argument to show that $\Sigma(\psi, y) \subset \Sigma(\phi, x)$ and hence $\Sigma(\phi, x) = \Sigma(\psi, y)$ as required.

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[3]Note that this lemma indicates that if $\sigma_\phi \subset \sigma_\psi$ then the Cauchy structure on $\mathcal{M}$ given by ψ is both larger than and compatible with the Cauchy structure on $\mathcal{M}$ given by ϕ.

Now, suppose that for all $x \in \partial(\phi(\mathcal{M}))$ there exists $y \in \partial(\psi(\mathcal{M}))$ so that $\Sigma(\phi, x) = \Sigma(\psi, y)$. Let $[x] \in \sigma_\phi$, where $x \in \partial(\phi(\mathcal{M}))$, and let $y \in \partial(\psi(\mathcal{M}))$ be such that $\Sigma(\phi, x) = \Sigma(\psi, y)$. Let $\mathfrak{q} \subset \mathcal{M}$ be a sequence so that $\phi(\mathfrak{q}) \to x$, then there exists $\mathfrak{s} \in \Sigma_0(\mathcal{M})$ so that $\mathfrak{s} \subset \mathfrak{q}$ and $\phi(\mathfrak{s}) \to x$. Then, by construction, $\psi(\mathfrak{s}) \to y$, so that $\psi(\mathfrak{q}) \to y$, and hence $y \rhd x$. Since $\Sigma(\phi, x) = \Sigma(\psi, y)$ we can use the same argument to show that $x \rhd y$ and therefore $[x] = [y]$. Thus $\sigma_\phi \subset \sigma_\psi$ as required. $\qquad\square$

Next we establish a collection of equivalent definitions of $\simeq$.

Proposition 3.5. *Let $\mathcal{M}$ be a manifold, let $\phi : \mathcal{M} \to \mathcal{M}_\phi$ and $\psi : \mathcal{M} \to \mathcal{M}_\psi$ be two envelopments of $\mathcal{M}$. The following are equivalent:*

1. *The envelopments ϕ and ψ are equivalent, $\phi \simeq \psi$.*

2. *There exists a homeomorphism, $f : \overline{\phi(\mathcal{M})} \to \overline{\psi(\mathcal{M})}$, so that $f\phi = \psi$.*

3. *For all $x \in \partial(\phi(\mathcal{M}))$ there exists $y \in \partial(\psi(\mathcal{M}))$ so that $\Sigma(\phi, x) = \Sigma(\psi, y)$ and for all $p \in \partial(\psi(\mathcal{M}))$ there exists $q \in \partial(\phi(\mathcal{M}))$ so that $\Sigma(\psi, p) = \Sigma(\phi, q)$.*

Proof. **3 $\Leftrightarrow$ 1** Apply lemma 3.4 twice.

1 $\Rightarrow$ 2 Since $\phi \simeq \psi$ we know that $\sigma_\phi = \sigma_\psi$. Therefore for each $x \in \partial(\phi(\mathcal{M}))$ there exists $y \in \partial(\psi(\mathcal{M}))$ so that $x \equiv y$; denote this y by x_ψ. Define $f : \overline{\phi \mathcal{M}} \to \overline{\psi \mathcal{M}}$ by

$$f(x) = \begin{cases} \psi \phi^{-1}(x) & \text{if } x \in \phi \mathcal{M} \\ x_\psi & \text{otherwise.} \end{cases}$$

Since $\psi \phi^{-1}$ is a homeomorphism, we need only consider $f|_{\partial(\phi(\mathcal{M}))}$.

We need to show that f is well defined. Suppose that there exist $u, v \in \partial(\psi(\mathcal{M}))$ so that $x \equiv u$ and $x \equiv v$, where $x \in \partial(\phi(\mathcal{M}))$. Since, $\equiv$ is an equivalence relation $u \equiv v$. By theorem 2.4 and as $\mathcal{M}_\psi$ is hausdorff, we can conclude that $u = v$. Therefore f is well defined.

We need to show that f is surjective. Let $y \in \partial(\psi(\mathcal{M}))$ then, since $\phi \simeq \psi$, there exists $x \in \partial(\phi(\mathcal{M}))$ so that $[y] = [x]$ and hence $f(x) = y$. Therefore f is surjective.

We need to show that f is injective. Suppose that $x, y \in \partial(\phi(\mathcal{M}))$ are such that $f(x) = f(y)$, then $x \equiv f(x) = f(y) \equiv y$ so that $x \equiv y$. As before by theorem 2.4 and as $\mathcal{M}_\phi$ is hausdorff, we know that $x = y$.

We need to show that f is continuous. Since $\mathcal{M}$ is first countable we can do this by showing that f is sequentially continuous. The proof that f is sequentially continuous is long. It is divided into five sections. In the first section we show that f is continuous on $\phi \mathcal{M}$. The second section shows that any for any sequence $\{x_i\} \subset \phi \mathcal{M}$ with unique limit point $x \in \partial(\phi(\mathcal{M}))$ we have that $\{f(x_i)\}$ converges to $f(x)$. The arguments of the section also shows that a similar statement holds for f^{-1}. The third section shows that for any sequence $\{x_i\} \subset \partial(\phi(\mathcal{M}))$ converging to x, necessarily in $\partial(\phi(\mathcal{M}))$, it

is the case that $f(x)$ is a limit point of $\{f(x_i)\}$. The fourth section shows that the sequence $\{f(x_i)\}$ of the third section uniquely converges to $f(x)$. The fifth section considers sequences in $\overline{\phi\mathcal{M}}$ whose elements are not restricted to either $\phi\mathcal{M}$ or $\partial\phi\mathcal{M}$. These arguments demonstrate that f is sequentially continuous and therefore continuous. Since we shall repeat the arguments of earlier paragraphs in later sections we will number all paragraphs to make reference to the arguments easier.

1 First, since f restricted to $\phi\mathcal{M}$ is $\psi\phi^{-1}$ we only need consider sequences that converge to points in $\partial(\phi(\mathcal{M}))$.

2 Second, suppose that $\{x_i\} \subset \phi\mathcal{M}$ converges to $x \in \partial((\phi))\mathcal{M}$. Since $f(x) \equiv x$ we know that $\{f(x_i)\}$ must have $f(x)$ as a limit point (by theorem 2.4). Any subsequence $\{p_i\}$ of $\{x_i\}$ must also be such that $\{f(p_i)\}$ has $f(x)$ as a limit point by the same reasoning, since $\{p_i\}$ must converge uniquely to x. We will show that $f(x)$ is unique. Suppose that there exists $q \in \overline{\psi\mathcal{M}}$ and a subsequence $\{q_i\}$ of $\{x_i\}$ so that $\{f(q_i)\}$ converges to q. Since $\{q_i\} \subset \{x_i\}$ we know that $\{q_i\}$ converges to x and as $f(x) \equiv x$ we know that $f(x)$ is an accumulation point of $\{f(q_i)\}$ but, by construction, $\{f(q_i)\}$ converges to q. Therefore $q = f(x)$. Thus for all sequences $\mathfrak{s}$ lying in $\phi\mathcal{M}$ so that $\mathfrak{s} \to x \in \partial\phi\mathcal{M}$ uniquely, we know that $f(\mathfrak{s}) \to f(x)$ uniquely. Since the argument of this paragraph can also be applied to f^{-1} we know that for all sequences $\mathfrak{s} \subset \psi\mathcal{M}$ so that $\mathfrak{s} \to y \in \partial\psi\mathcal{M}$ uniquely, we have that $f^{-1}(\mathfrak{s}) \to f^{-1}(y)$ uniquely. We use these facts below.

3 Third, suppose that $\{x_i\}$ is a sequence in $\partial(\phi(\mathcal{M}))$ converging to x and suppose that $\{f(x_i)\}$ has no limit points, then we can choose an open neighbourhood V of $f(x)$ so that for all i, $f(x_i) \notin V$. For each i choose a sequence $\{f(y_j^i)\} \subset \psi\mathcal{M}$ that converges uniquely to $f(x_i)$. As $\{f(y_j^i)\}$ converges to $f(x_i)$ we know that $\{f(y_j^i)\} \cap V$ must be finite, hence without loss of generality we may assume that for all i, j, $\{f(y_j^i)\} \cap V = \varnothing$ and therefore that for all i, j, $f(y_j^i) \notin V$. Using the same techniques as in paragraph **2** we can show that each $\{y_j^i\}$ must converge uniquely to x_i. Form a new sequence, $\{s_k\} = \bigcup_{i,j}\{y_j^i\}$. By construction, since each $\{y_j^i\}$ converges to x_i, we know that $\{s_k\}$ has x as a limit point and as $\{s_k\} \subset \phi\mathcal{M}$ we know that $\{f(s_k)\}$ has $f(x)$ as a limit point. This implies that there exists an infinite subsequence, $\{f(q_l)\}$ of $f(s_k)$ so that for all l, $f(q_l) \in V$. This is a contradiction, however, as for each l there exists i_l and j_l so that $q_l = y_{j_l}^{i_l}$ where we know that $f(q_l) \in V$ and that $f(y_{j_l}^{i_l}) \notin V$. Therefore $\{f(x_i)\}$ must have $f(x)$ as a limit point.

4 Fourth, suppose that $\{f(x_i)\}$ also has a limit point $q \in \partial(\phi(\mathcal{M}))$, where $\{x_i\}$ is the sequence of paragraph **3**. We may choose a sequence of open neighbourhoods, V_i, so that $x_i \in V_i$, for all $j \neq i$, $x_j \notin V_i$ and for all i, j, $i \neq j$, $\overline{V_i} \cap \overline{V_j} = \varnothing$. Let $\{y_j^i\} \subset \phi\mathcal{M}$ be a sequence that converges uniquely to x_i and is such that for all j, $y_j^i \in V_i$. Let $\{s_k\} = \bigcup_{i,j}\{y_j^i\}$ be a new sequence formed from the union of the y_j^i's. Since $\{f(x_i)\}$ has q as a limit point we know that $\{f(s_k)\}$ must also have q as a limit point. From paragraph **2** we know that $f^{-1}(q)$ must be a limit point of $\{s_k\}$. By construction this implies that $f^{-1}(q)$ is either equal to x_i for some i or equal to x.

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5 If $f^{-1}(q) = x$ then we are done, so suppose that there exists l so that $q = f(x_l)$. Since q is a limit point of $\{f(x_i)\}$ we can choose a subsequence, $\{q_r\}$ of $\{f(s_k)\}$ so that $q_r \in \{f(y_j^r)\}$ and $\{q_r\}$ uniquely converges to q. From paragraph **2** we know that $\{f^{-1}(q_r)\}$ must have $f^{-1}(q) = x_l$ as a unique limit point. This implies that $\{f^{-1}(q_r)\} \cap V_l$ must be infinite. But by construction we know that for all $r \neq l$, $y_j^r \notin V_l$, and since $q_r = y_j^r$ for some j we know that $\{f^{-1}(q_r)\} \cap V_l$ is either empty or contains only the element $f^{-1}(q_l)$. Therefore we have a contradiction and $q = f(x)$ as required.

6 Fifth, suppose that $\{x_i\}$ lies in $\overline{\phi\mathcal{M}}$ and that it uniquely converges to $x \in \partial(\phi(\mathcal{M}))$. If either $\{x_i\} \cap \partial(\phi(\mathcal{M}))$ or $\{x_i\} \cap \phi\mathcal{M}$ is finite we can use the arguments above to show that $\{f(x_i)\}$ uniquely converges to $f(x)$, so suppose that neither set is finite. In this case we know that both sequences $f(\{x_i\} \cap \partial(\phi(\mathcal{M})))$ and $f(\{x_i\} \cap \phi\mathcal{M})$ must uniquely converge to $f(x)$, and therefore $\{f(x_i)\}$ must also uniquely converge to $f(x)$. Hence f is continuous.

Since $x \rhd f(x)$, $\phi\psi^{-1}$ is continuous and f^{-1} is bijective the same argument can be applied to show that f^{-1} is continuous and therefore f is a homeomorphism.

2 $\Rightarrow$ 3 Suppose that there exists a homeomorphism, $f : \overline{\phi\mathcal{M}} \to \overline{\psi\mathcal{M}}$ so that $f\phi = \psi$. Let $x \in \partial(\phi(\mathcal{M}))$ and $y = f(x)$. Since $f\phi = \psi$, it is clear that $y \in \partial(\psi(\mathcal{M}))$. Let $\mathfrak{s} \in \Sigma(\phi, x)$. By the continuity of f, $\mathfrak{s} \in \Sigma(\psi, y)$, thus $\Sigma(\phi, x) \subset \Sigma(\psi, y)$.

Now, let $\mathfrak{t} \in \Sigma(\psi, y)$. By the continuity of f^{-1}, $\mathfrak{t} \in \Sigma(\phi, x)$. Hence $\Sigma(\psi, y) \subset \Sigma(\phi, x)$.

It follows that $\Sigma(\phi, x) = \Sigma(\psi, y)$. By a similar argument, it can be shown that for all $p \in \partial(\psi(\mathcal{M}))$ there exists $q \in \partial(\phi(\mathcal{M}))$ so that $\Sigma(\psi, p) = \Sigma(\phi, q)$. $\qquad \square$

The technique employed in **1 $\Rightarrow$ 2** can be very useful when working with the Abstract Boundary.

Corollary 3.6. *Let $\phi : \mathcal{M} \to \mathcal{M}_\phi$ and $\psi : \mathcal{M} \to \mathcal{M}_\psi$ be envelopments, let $x \in \partial(\phi(\mathcal{M})), y \in \partial(\psi(\mathcal{M}))$ be such that $x \equiv y$ and let $U \subset \mathcal{M}$. Then $x \in \overline{\phi(U)}$ if and only if $y \in \overline{\psi(U)}$.*

Proof. Suppose that $x \in \overline{\phi(U)}$. Then there exists a sequence $\{p_i\} \subset U$ so that $\{\phi(p_i)\} \to x$. From theorem 2.4 we know that $\{\psi(p_i)\}$ must have y as a limit point. Since $\{\psi(p_i)\} \subset \psi(U)$ then $y \in \overline{\psi(U)}$. The same argument can be applied in the reverse direction. $\qquad \square$

3.1 Cauchy structures

Note that the proof of proposition 3.5 is very similar to the proof that uniformly continuous functions $f : X \to Y$ into a complete space Y extend uniquely to the completion of X. This similarity is not accidental, even though in our case the functions involved are not necessarily uniformly continuous. The similarity exists because there is a more general

extension theorem (see [13]) regarding Cauchy continuous functions which does apply in our case.

The more general theorem is best expressed in the language of Cauchy spaces [11]. Since the majority of our intended audience are unlikely to be familiar with Cauchy spaces and as explicit proofs are instructive, we have given full proofs of the main results, propositions 3.5 and 4.8, without appealing to Cauchy spaces. For those readers who are familiar we include below a brief review of the needed material, the relevant extension theorem and the alternative proof of proposition 3.5. At the end of section 4 we give an alternative proof of proposition 4.8.

Cauchy structures on a set are usually defined in terms of filters. In any first countable topological space, however, filters are equivalent to sequences. We provide the following definition:

Definition 3.7. *A Cauchy structure on a metric space T is a subset S of the set $\Sigma(T)$ of all sequences in T so that the set of filters corresponding to the sequences in S generates a Cauchy structure in the sense of [11]. That is, the set $\mathcal{F}$ of filters corresponding to the sequences in S is such that*

1. for each $x \in T$ the ultrafilter at x is in $\mathcal{F}$,

2. if $F \in \mathcal{F}$ and $F \subset G$, where G is a filter, then $G \in \mathcal{F}$,

3. if $F, G \in \mathcal{F}$ and every element of F intersects every element of G then $F \cap G \in \mathcal{F}$.

We say that a Cauchy structure S on a metric space T is compatible with the topological structure if, for every $\mathfrak{s} \in S$, we have that either $\mathfrak{s}$ has no limit points or every subsequence $\mathfrak{q}$ of $\mathfrak{s}$ has a unique limit point with respect to the topology. Whenever we impose a Cauchy structure on a metric space we will implicitly assume that the topology and Cauchy structure are compatible.

We have two specific situations to consider. Let d be a distance on a manifold $\mathcal{M}$, then the set of all sequences that are Cauchy with respect d is a Cauchy structure. Let $\phi : \mathcal{M} \to \mathcal{M}_\phi$ be an envelopment of a manifold $\mathcal{M}$, then the set of all sequences $\mathfrak{s}$ in $\mathcal{M}$ so that every subsequence $\mathfrak{q} \subset \mathfrak{s}$ converges under ϕ is a Cauchy structure.

Definition 3.8. *A function $f : X \to Y$ between metric spaces is Cauchy continuous with respect to the Cauchy structures C_X and C_Y on X and Y respectively if, for all $\mathfrak{s} \in C_X$, $f(\mathfrak{s}) \in C_Y$.*

Definition 3.9. *Let Y be a metric space with Cauchy structure S. Then Y is said to be complete with respect to S if and only if every sequence $\mathfrak{s} \in S$ converges to a point with respect to the topology on Y. Where the selection of a Cauchy structure S is clear, we shall simply say that Y is complete.*

Definition 3.10. *Let X and Y be metric spaces. Then Y is a completion of X if Y is complete and there exists $f : X \to Y$, a Cauchy continuous map, so that $f(X)$ is dense in Y.*

In particular, if $\phi : \mathcal{M} \to \mathcal{M}_\phi$ is an envelopment then $\mathcal{M}$ carries the Cauchy structure derived from ϕ discussed above. With respect to this structure, the topological space $\overline{\phi(\mathcal{M})}$ is a completion of $\mathcal{M}$. Likewise, the usual Cauchy completion of a manifold is a completion, in the sense above, with respect to the Cauchy structure induced by a distance and the inclusion embedding.

We have the following result:

Theorem 3.11. *Let Y be a metric space which is a completion of the metric space X, with respect to the Cauchy continuous map $k : X \to Y$ and the Cauchy structures C_X and C_Y on X and Y respectively. Let Z be a metric space complete with respect to the Cauchy structure C_Z on Z. Then a Cauchy continuous function $f : X \to Z$ has a unique continuous and Cauchy continuous extension $\hat{f} : Y \to Z$ so that $\hat{f} \circ k = f$.*

Proof. Refer to [13] or [11]. □

We are now able to give our alternate proof of proposition 3.5.

Alternative proof of proposition 3.5. **3 $\Leftrightarrow$ 1** Apply lemma 3.4 twice.

1 $\Rightarrow$ 2 Define $f : \mathcal{M} \to \overline{\psi(\mathcal{M})}$ by $f(x) = \psi(x)$ for all $x \in \mathcal{M}$. Equip $\mathcal{M}$ and $\overline{\psi(\mathcal{M})}$ with the Cauchy structures C_ϕ and C_ψ induced by inclusion into $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$, respectively. That is, a sequence $\mathfrak{s}$ is Cauchy in $\mathcal{M}$ if and only if $\phi(\mathfrak{s})$ converges in $\overline{\phi(\mathcal{M})}$ and a sequence $\mathfrak{s}$ is Cauchy in $\overline{\psi(\mathcal{M})}$ if and only if $\mathfrak{s}$ converges in $\overline{\psi(\mathcal{M})}$.

We will now show that f is Cauchy continuous. Let $\mathfrak{s} \in C_\phi$. If $\phi(\mathfrak{s})$ converges in $\phi(\mathcal{M})$ then $\psi(\mathfrak{s})$ will also converge in $\psi(\mathcal{M})$, by the continuity of $\psi \circ \phi^{-1}$. So suppose then that $\phi(\mathfrak{s}) \to x \in \partial(\phi(\mathcal{M}))$ uniquely. By lemma 3.4 there exists $y \in \partial(\psi(\mathcal{M}))$ so that $\Sigma(\phi, x) = \Sigma(\psi, y)$. Thus, as $\mathfrak{s} \in \Sigma(\phi, x)$, we know that $\mathfrak{s} \in \Sigma(\psi, y)$. Hence $\psi(\mathfrak{s})$ has y as a limit point. To show that $f(\mathfrak{s})$ is Cauchy, however, we must show that the limit point of $\psi(\mathfrak{s})$ is unique.

Let us first suppose that $\psi(\mathfrak{s})$ has a subsequence $\psi(\mathfrak{p})$ with no limit points. Since $\mathfrak{p} \subset \mathfrak{s}$ we know that $\mathfrak{p} \in \Sigma(\phi, x)$ and therefore that $\mathfrak{p} \in \Sigma(\psi, y)$. This contradicts our assumption that $\psi(\mathfrak{p})$ has no limit points. Likewise suppose that $\psi(\mathfrak{s})$ also has $p \in \overline{\psi(\mathcal{M})}$, $p \neq y$, as a limit point. Then there must exist some subsequence $\psi(\mathfrak{p})$ of $\psi(\mathfrak{s})$ so that $\psi(\mathfrak{p}) \to p$ uniquely. Since $\mathfrak{p} \subset \mathfrak{s}$, we know that $\mathfrak{p} \in \Sigma(\phi, x)$ and hence $\mathfrak{p} \in \Sigma(\psi, y)$. Once again this is a contradiction. It follows that $\psi(\mathfrak{s})$ converges to y in $\partial(\psi(\mathcal{M}))$ and so $\psi(\mathfrak{s}) \in C_\psi$. Therefore the function f is Cauchy continuous.

Thus, by theorem 3.11, f has a unique extension $\hat{f} : \overline{\phi(\mathcal{M})} \to \overline{\psi(\mathcal{M})}$ so that $\hat{f}\phi = \psi$. By the uniqueness of $\hat{f}$ and as the argument is symmetric in ϕ and ψ, it must be the case that $\hat{f}$ is a homeomorphism.

2 $\Rightarrow$ 3 Suppose that there exists a homeomorphism, $f : \overline{\phi(\mathcal{M})} \to \overline{\psi(\mathcal{M})}$ so that $f\phi = \psi$. Let $x \in \partial(\phi(\mathcal{M}))$ and $y = f(x)$. Since $f\phi = \psi$, it is clear that $y \in \partial(\psi(\mathcal{M}))$. Let $\mathfrak{s} \in \Sigma(\phi, x)$. By the continuity of f, $\mathfrak{s} \in \Sigma(\psi, y)$, thus $\Sigma(\phi, x) \subset \Sigma(\psi, y)$. Now, let $\mathfrak{t} \in \Sigma(\psi, y)$. By the continuity of f^{-1}, $\mathfrak{t} \in \Sigma(\phi, x)$. Hence $\Sigma(\psi, y) \subset \Sigma(\phi, x)$.

It follows that $\Sigma(\phi, x) = \Sigma(\psi, y)$. By a similar argument, it can be shown that for all $p \in \partial(\psi(\mathcal{M}))$ there exists $q \in \partial(\phi(\mathcal{M}))$ so that $\Sigma(\psi, p) = \Sigma(\phi, q)$.

$\square$

4 An equivalence on a class of distances

We now have a way to describe when two envelopments produce the same Abstract Boundary points in terms of sequences in $\mathcal{M}$. We present here an equivalence relation on a set of distances on $\mathcal{M}$. This relation mirrors the one in the previous section and will allow us, in the next section, to define a one-to-one correspondence between the two sets of equivalence classes.

First we need to define the structures with which we will be working.

Definition 4.1. *Let $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be a distance. Let $\mathcal{C}(d) = \{\mathfrak{s} \in \Sigma_0(\mathcal{M}) : \mathfrak{s}$ is Cauchy with respect to $d\}$[4]*

Definition 4.2. *Let $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be a distance. Let $\mathcal{M}^d$ denote the Cauchy completion of $\mathcal{M}$ with respect to d. We think of $\mathcal{M}^d$ as the set of all Cauchy sequences in $\mathcal{M}$ under an equivalence relation. Two Cauchy sequences $\mathfrak{u}$ and $\mathfrak{v}$ in $\mathcal{M}$ are equivalent if $\lim_{i,j\to\infty} d(u_i, v_j) = 0$[5]. Let $\mathfrak{u}$ be a Cauchy sequence in $\mathcal{M}$, then we shall denote the equivalence class of $\mathfrak{u}$ by $[\mathfrak{u}]_d$. Where unambiguous we will drop the subscript and simply write $[\mathfrak{u}]$. We topologise $\mathcal{M}^d$ by extending the distance d to $\mathcal{M}^d$. Let $d^* : \mathcal{M}^d \times \mathcal{M}^d \to \mathbb{R}$ be such that $d^*([\mathfrak{u}], [\mathfrak{v}]) = \lim_{i,j\to\infty} d(u_i, v_j)$. Then there exists an isometry $\imath_d : \mathcal{M} \to \mathcal{M}^d$ so that $\imath_d(x) = [\mathfrak{w}_x]$ where $\mathfrak{w}_x$ is the constant sequence at x. Note that, in general, $\mathcal{M}^d$ is not necessarily a manifold with boundary.*

For utility, we give the following trivial result.

Lemma 4.3. *Let $\mathfrak{u} = \{u_i\}$ be a Cauchy sequence in $\mathcal{M}$ with respect to d, then $\{\imath_d(u_i)\} \to [\mathfrak{v}]$ if and only if $\mathfrak{u} \in [\mathfrak{v}]$.*

Proof. Suppose that $\{\imath_d(u_i)\}_i \to [\mathfrak{v}]$. Then for all $\epsilon > 0$ there exists $i^* \in \mathbb{N}$ so that for all $i > i^*$, $d^*(\imath_d(u_i), [\mathfrak{v}]) < \epsilon$. The condition $d^*(\imath_d(u_i), [\mathfrak{v}]) < \epsilon$ is equivalent to $\lim_{i,j\to\infty} d(u_i, v_j) < \epsilon$, however, and so $\lim_{i,j\to\infty} d(u_i, v_j) = 0$. Thus $\mathfrak{u} \in [\mathfrak{v}]$.

Suppose that $\mathfrak{u} \in [\mathfrak{v}]$. Then $\lim_{i,j\to\infty} d(u_i, v_j) = 0$, so that for all $\epsilon > 0$ there exists $i^* \in \mathbb{N}$ so that if $i > i^*$, then $\lim_{i,j\to\infty} d(u_i, v_j) < \epsilon$. As before this implies that $d^*(\imath_d(u_i), [\mathfrak{v}]) < \epsilon$, from which we can see that $\{\imath_d(u_i)\}_i \to [\mathfrak{v}]$. $\square$

Now we define an equivalence relation on the set of all distances on $\mathcal{M}$.

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[4]The set $C(d)$ captures the portion of the Cauchy structure given by d which is relevant for the a-boundary.

[5]As $\mathfrak{u}$ and $\mathfrak{v}$ are Cauchy sequences the same limit will be achieved regardless of the choice of ordering of each sequence.

Definition 4.4. *Let d and d' be distances on $\mathcal{M}$. Then d and d' are equivalent, $d \simeq d'$, if and only if $\mathcal{C}(d) = \mathcal{C}(d')$*[6].

At some point we must link distances with envelopments, as that is the aim of this paper. Unfortunately, it is reasonably easy to give examples of distances on a manifold that have no relation to any envelopments[7].

Definition 4.5. *Let $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be a distance on $\mathcal{M}$ and define $E(d)$ to be the set of all non-trivial envelopments $\phi : \mathcal{M} \to \mathcal{M}_\phi$ so that there exists a complete distance $d' : \mathcal{M}_\phi \times \mathcal{M}_\phi \to \mathbb{R}$ with the property that $d'|_{\phi(\mathcal{M}) \times \phi(\mathcal{M})} \simeq d$.*

Definition 4.6. *Let $D(\mathcal{M}) = \{d : E(d) \neq \varnothing\}$.*

It is interesting that $D(\mathcal{M}) = \varnothing$ is equivalent to the compactness of $\mathcal{M}$.

Proposition 4.7. *Let $\mathcal{M}$ be a manifold. The set $D(\mathcal{M})$ is empty if and only if $\mathcal{M}$ is compact.*

Proof. $\Leftarrow$ Since $\mathcal{M}$ is compact there are no non-trivial envelopments of $\mathcal{M}$. Hence for all distances d on $\mathcal{M}$, $E(d) = \varnothing$. Therefore for all d, $d \notin D(\mathcal{M})$ and $D(\mathcal{M})$ must be empty.

$\Rightarrow$ Suppose that $\mathcal{M}$ is not compact. Then there exists a sequence $\mathfrak{s}$ in $\mathcal{M}$ that has no limit points in $\mathcal{M}$. By the Endpoint Theorem (see [1]), we may use this sequence to construct an envelopment, $\phi : \mathcal{M} \to \mathcal{M}_\phi$ so that $\phi(\mathfrak{s})$ converges to some point on $\partial(\phi(\mathcal{M}))$.

Choose a distance $d_\phi : \mathcal{M}_\phi \times \mathcal{M}_\phi \to \mathbb{R}$ so that d_ϕ is complete. Define the distance $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ by $d(x,y) = d_\phi(\phi(x), \phi(y))$. Then $d \simeq d_\phi|_{\phi(\mathcal{M}) \times \phi(\mathcal{M})}$ so that $\phi \in E(d)$ and hence $d \in D(\mathcal{M})$.

Taking the contrapositive of this result we see that $D(\mathcal{M}) = \varnothing$ implies that $\mathcal{M}$ is compact. $\square$

Just as for envelopments, there are several different ways to express the relation $\simeq$; again we give only those that are useful.

Proposition 4.8. *Let d and d' be distances on $\mathcal{M}$, then $d \simeq d'$ if and only if there exists a homeomorphism $f : \mathcal{M}^d \to \mathcal{M}^{d'}$ so that $f\iota_d = \iota_{d'}$.*

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[6]Note that this is slightly different from Cauchy equivalence since we are only concerned with Cauchy sequences that have no limit points in $\mathcal{M}$.

[7]For example, let $\mathcal{M} = \mathbb{R}^+$ with the embedding given by $\phi(x) = (x, \sin(\frac{1}{x}))$ into $\mathbb{R}^2$. Then the usual euclidean distance on $\mathbb{R}^2$ induces the distance $d(x,y) = \sqrt{(x-y)^2 + (\sin\frac{1}{x} - \sin\frac{1}{y})^2}$ on $\mathcal{M}$. We note that the sequences $\{x_i = \frac{1}{2i\pi}\}$, $\{y_i = \frac{2}{4i\pi + \pi}\}$ and $\{z_i = \frac{2}{4i\pi + 3\pi}\}$ are all Cauchy with respect to d. Thus for d to be induced by an envelopment ϕ it must be the case that the sequences $\{\phi(x_i)\}$, $\{\phi(y_i)\}$ and $\{\phi(z_i)\}$ each have a unique limit point. Moreover any envelopment of $\mathcal{M}$ must be into either $\mathbb{R}$ or S^1. It will, therefore, have either zero, one or two boundary points. This implies that at least one of the pairs of sequences $(\{\phi(x_i)\}, \{\phi(y_i)\})$, $(\{\phi(x_i)\}, \{\phi(z_i)\})$ and $(\{\phi(y_i)\}, \{\phi(z_i)\})$ must share the same limit point. We can calculate, however, that $\lim_{i \to \infty} d(x_i, y_i) = 1$, $\lim_{i \to \infty} d(x_i, z_i) = 1$ and $\lim_{i \to \infty} d(y_i, z_i) = 2$. This implies that none of the sequences $\{\phi(x_i)\}, \{\phi(y_i)\}, \{\phi(z_i)\}$ share a limit point. As this is a contradiction, d cannot be induced by any envelopment ϕ. Hence we must restrict the set of distances in which we are interested.

Proof. The proof of this result follows the same format as the proof of step $\mathbf{1} \Rightarrow \mathbf{2}$ in proposition 3.5.

$\Rightarrow$

Since $d \simeq d'$, we know that $\mathcal{C}(d) = \mathcal{C}(d')$, and hence we can define $f : \mathcal{M}^d \to \mathcal{M}^{d'}$ by $f([\mathfrak{s}]_d) = [\mathfrak{s}]_{d'}$[8] where $\mathfrak{s}$ is Cauchy with respect to d. Note that $\mathfrak{s}$ may have a limit point in $\mathcal{M}$, and hence would not be in $\Sigma_0(\mathcal{M})$.

Because $\mathcal{C}(d) = \mathcal{C}(d')$ we can easily see that f is surjective. If $f([\mathfrak{u}]) = f([\mathfrak{v}])$ then the sequence $\mathfrak{w}$ given by;

$$ w_i = \begin{cases} u_{\frac{i}{2}} & \text{iff } i = 2n \\ v_{\frac{i+1}{2}} & \text{iff } i = 2n+1 \end{cases} $$

is such that $\mathfrak{w} \in \mathcal{C}(d')$. Therefore $\mathfrak{w} \in \mathcal{C}(d)$ and $[\mathfrak{u}]_d = [\mathfrak{v}]_d$. We can conclude that f is bijective and by definition we know that $f\imath_d = \imath_{d'}$. Thus f is continuous on $\imath_d\mathcal{M}$.

Since $\mathcal{M}^d$ is first countable to show that f is continuous everywhere it is enough to show that f is sequentially continuous. Since f is continuous on $\imath_d\mathcal{M}$, we need only consider sequences in $\mathcal{M}^d$ that have limit points in $\partial(\imath_d(\mathcal{M}))$.

The proof that f is sequentially continuous is long. We have divided the proof into four sections. The first section shows that for any sequence $\mathfrak{s} \subset \imath_d\mathcal{M}$ so that $\mathfrak{s} \to x \in \partial\imath_d\mathcal{M}$ uniquely we have that $f(\mathfrak{s}) \to f(x)$ uniquely. The second section show that if $\mathfrak{s} \subset \partial\imath_d\mathcal{M}$ converges uniquely to $x \in \partial\imath_d\mathcal{M}$ then $f(\mathfrak{s})$ has $f(x)$ as a, not necessarily unique, limit point. The third section demonstrates that for $\mathfrak{s}$ as in the second section the limit point $f(x)$ of $f(\mathfrak{s})$ is unique. The fourth section considers sequences in $\mathcal{M}^d$ without restriction to either $\imath_d\mathcal{M}$ or $\partial\imath_d\mathcal{M}$. We shall sometimes repeat the arguments of an earlier section in later portions of the proof. To aid reference we have numbered the paragraphs.

1 First, let $\{x_i\}$ be a sequence in $\imath_d\mathcal{M}$ that converges uniquely to $x \in \partial(\imath_d(\mathcal{M}))$. Since $x_i \in \imath_d\mathcal{M}$ there must exist $y_i \in \mathcal{M}$ so that $\imath_d(y_i) = x_i$. By lemma 4.3 we know that $x = [\{y_i\}]_d$. Therefore $f(x) = f([\{y_i\}]_d) = [\{y_i\}]_{d'}$. Once again by lemma 4.3 we know that $\{\imath_{d'}(y_i)\}$ converges uniquely to $[\{y_i\}]_{d'} = f(x)$. But $f(x_i) = f(\imath_d(y_i)) = \imath_{d'}(y_i)$, and therefore $f(x_i) \to f(x)$ as required. Thus for all sequences $\mathfrak{s}$ lying in $\imath_d\mathcal{M}$ so that $\mathfrak{s} \to x \in \partial(\imath_d(\mathcal{M}))$ uniquely we know that $f(\mathfrak{s}) \to f(x)$ uniquely. Since the argument of this paragraph can also be applied to f^{-1} we know that for all sequences $\mathfrak{s} \subset \imath d'\mathcal{M}$ so that $\mathfrak{s} \to t \in \partial(\imath_{d'}(\mathcal{M}))$ uniquely, we have that $f^{-1}(\mathfrak{s}) \to f^{-1}(y)$ uniquely. We use these facts below.

2 Second, let $\{x_i\}$ be a sequence in $\partial(\imath_d(\mathcal{M}))$ that converges uniquely to $x \in \partial(\imath_d(\mathcal{M}))$ and suppose that $\{f(x_i)\}$ has no limit points in $\mathcal{M}^{d'}$. We may choose an open neighbourhood V of $f(x)$ so that for all i, $f(x_i) \notin V$. For each i we can choose a sequence $\{f(\imath_d y_j^i)\}$ converging uniquely to $f(x_i)$. This implies that $\{f(\imath_d y_j^i)\} \cap V$ must be finite and hence, without loss of generality, we can assume that $\{f(\imath_d y_j^i)\} \cap V = \varnothing$ and therefore that for all i, j, $f(\imath_d y_j^i) \notin V$. From paragraph **1** we can conclude that for each i the sequence $\{\imath_d y_j^i\}$ must converge uniquely to x_i. Now, by construction, for the sequence $\mathfrak{s} = \bigcup_{i,j}\{\imath_d(y_j^i)\}$, we can conclude that $f(\mathfrak{s})$ does not have $f(x)$ as a limit point. The sequence $\mathfrak{s}$ has x_i as a

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[8]This is the unique f given by the lifting of $\imath_d$ to $\mathcal{M}^{d'}$.

limit point for each i and as $\{x_i\} \to x$ we know that $\mathfrak{s}$ has x as a limit point. Choose a subsequence, $\mathfrak{p} = \{\imath_d(p_k)\}$ of $\mathfrak{s}$ so that $p_k \in \mathcal{M}$ and $\mathfrak{p} \to x$. Hence, from above, we know that $x = [\{p_k\}]_d$ and that $\{f(\imath_d p_k) = \imath_{d'}(p_k)\} \to [\{p_k\}]_{d'} = f(x)$. This is a contradiction, since $\{f(\imath_d p_k)\} \to f(x)$ implies that $f(\mathfrak{p}) \cap V \neq \varnothing$, but $\mathfrak{p} \subset \mathfrak{s}$ and $f(\mathfrak{s}) \cap V = \varnothing$. Therefore $\{f(x_i)\}$ has $f(x)$ as a limit point.

3 Third, we will now show that $f(x)$ is the unique limit point of $\{f(x_i)\}$, where $\{x_i\}$ and x are the sequence and point of paragraph **2**. Suppose that $q \in \mathcal{M}^{d'}$ is a limit point of $\{f(x_i)\}$; since $\{x_i\} \subset \partial \imath_d \mathcal{M}$ and $\{f(x_i)\} \subset \partial \imath_{d'} \mathcal{M}$ we know that $q \in \partial \imath_{d'} \mathcal{M}$. We shall show that $q = f(x)$. Since $\{x_i\}$ converges to x uniquely, we may choose a sequence of open neighbourhoods, V_i, so that $x_i \in V_i$, for all $j \neq i$, $x_j \notin V_i$ and for all i, j, $i \neq j$, $\overline{V_i} \cap \overline{V_j} = \varnothing$. Let $\{\imath_d y_j^i\} \subset \phi \mathcal{M}$ be a sequence that converges uniquely to x_i and is such that for all j, $\imath_d y_j^i \in V_i$. Let $\mathfrak{s} = \bigcup_{i,j} \{\imath_d y_j^i\}$ be a new sequence formed from the union of the $\{\imath_d y_j^i\}$'s. Since $\{f(x_i)\}$ has q as a limit point we know that $\{f(\mathfrak{s})\}$ must also have q as a limit point. From paragraph **1** we know that $f^{-1}(q)$ must be a limit point of $\mathfrak{s}$. By construction this implies that $f^{-1}(q)$ is either equal to x_i for some i or equal to x.

4 If $f^{-1}(q) = x$ then we are done, so suppose that there exists l so that $q = f(x_l)$. Since q is a limit point of $\{f(x_i)\}$ we can choose a subsequence, $\mathfrak{q} = \{q_k\}$ of $\{f(\mathfrak{s})\}$ so that $q_r \in \{f(\imath_d y_j^r)\}$ and $\mathfrak{q}$ uniquely converges to q. From paragraph **1** we know that $\{f^{-1}(q_r)\}$ must have $f^{-1}(q) = x_l$ as a unique limit point. This implies that $\{f^{-1}(q_r)\} \cap V_l$ must be infinite. But by construction we know that for all $r \neq l$, $y_j^r \notin V_l$, and since $q_r = y_j^r$ for some j we know that $\{f^{-1}(q_r)\} \cap V_l$ is either empty or contains only the element $f^{-1}(q_l)$. Therefore we have a contradiction and $q = f(x)$ as required. Thus for all sequences $\mathfrak{s}$ lying in $\partial(\imath_d(\mathcal{M}))$ with the unique limit point x we know that $f(\mathfrak{s})$ has the unique limit point $f(x)$ as required. Since the argument of the last three paragraphs applies to f^{-1} we also know that for all sequences $\mathfrak{s} \subset \imath_{d'} \mathcal{M}$ converging to $x \in \partial(\imath_{d'}(\mathcal{M}))$ that $f^{-1}(\mathfrak{s}) \to f^{-1}(x)$.

5 Fourth, suppose that $\mathfrak{x}$ is a sequence in $\mathcal{M}^d$ so that $\mathfrak{x} \to x \in \partial \imath_d \mathcal{M}$. Suppose that $\mathfrak{x} \cap \partial \imath_d \mathcal{M}$ is finite and let $\mathfrak{w}$ be the sequence given by $\mathfrak{x} \cap \imath_d \mathcal{M}$. Then from above we know that $f(\mathfrak{w}) \to f(x)$ uniquely. Since $\mathfrak{x} - \mathfrak{w}$ is finite we also know that $f(\mathfrak{x})$ must converge uniquely to $f(x)$. We can use the same technique to show that $f(\mathfrak{x}) \to f(x)$ uniquely if $\mathfrak{x} \cap \imath_d \mathcal{M}$ is finite. So suppose that $\mathfrak{x} \cap \imath_d \mathcal{M}$ and $\mathfrak{x} \cap \partial \imath_d \mathcal{M}$ are infinite and let $\mathfrak{u} = \mathfrak{x} \cap \imath_d \mathcal{M}$ and $\mathfrak{v} = \mathfrak{x} \cap \partial \imath_d \mathcal{M}$. From above we know that $f(\mathfrak{u}) \to f(x)$ uniquely and that $f(\mathfrak{v}) \to f(x)$ uniquely, therefore $f(\mathfrak{x}) = f(\mathfrak{u}) \cup f(\mathfrak{v})$ is such that $f(\mathfrak{x}) \to f(x)$ uniquely.

The continuity of f^{-1} follows by symmetry.

$\Leftarrow$

Let $\{x_i\} \in \mathcal{C}(d)$. Then $[\{x_i\}]_d \in \mathcal{M}^d$ and the sequence $\{\imath_d(x_i)\}_i$ converges uniquely to $[\{x_i\}]_d$. Since $f(\imath_d(x_i)) = \imath_{d'}(x_i)$ and as f is a homeomorphism we know that $\{\imath_{d'}(x_i)\}_i$ must uniquely converge to $[\{x_i\}]_{d'}$. Thus $\{x_i\} \in \mathcal{C}(d')$.

By symmetry we can conclude that $\mathcal{C}(d) = \mathcal{C}(d')$ and that $d \simeq d'$ as required. $\qquad\square$

This gives a useful corollary.

Corollary 4.9. *Let $d \in D(\mathcal{M})$, then there exists a homeomorphism $f : \mathcal{M}^d \to \overline{\phi(\mathcal{M})}$ so that $f \imath_d = \phi$, where $\phi \in E(d)$.*

Proof. Since $\phi \in E(d)$ there exists d_ϕ, a complete distance on $\mathcal{M}_\phi$, so that $d_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})} \simeq d$. Let $d' = d_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})}$. From proposition 4.8 we know that there exists a homeomorphism $h : \mathcal{M}^d \to \mathcal{M}^{d'}$ where $h\imath_d = \overline{\imath_{d'}}$.

As d_ϕ is complete we know that $\overline{\phi(\mathcal{M})}$ is homeomorphic to $\mathcal{M}^{d'}$, and by the universality of the Cauchy completion we know that there exists a homeomorphism $g : \mathcal{M}^{d'} \to \overline{\phi(\mathcal{M})}$, so that $g\imath_{d'} = \phi$.

Let $f : \mathcal{M}^d \to \overline{\phi(\mathcal{M})}$ be defined by $f = gh$. Then f is a homeomorphism and $f\imath_d = gh\imath_d = g\imath_{d'} = \phi$ as required. $\square$

Proposition 4.10. *Let $d, d' \in D(\mathcal{M})$, then $d \simeq d'$ if and only if $E(d) = E(d')$.*

Proof. $\Rightarrow$ Let $\phi : \mathcal{M} \to \mathcal{M}_\phi$ be an element of $E(d)$, then there exists a complete distance $d_\phi : \mathcal{M}_\phi \times \mathcal{M}_\phi \to \mathbb{R}$ so that $d_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})} \simeq d$. Since $d \simeq d'$ and $\simeq$ is an equivalence relation, we can see that $d_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})} \simeq d'$. Hence $\phi \in E(d')$. By symmetry we can conclude that $E(d) = E(d')$.

$\Leftarrow$ Since $E(d) = E(d')$ we can choose $\phi : \mathcal{M} \to \mathcal{M}_\phi$ an envelopment so that there exists two complete distances

$$d_\phi : \mathcal{M}_\phi \times \mathcal{M}_\phi \to \mathbb{R}$$

and

$$d'_\phi : \mathcal{M}_\phi \times \mathcal{M}_\phi \to \mathbb{R}$$

so that

$$d_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})} \simeq d$$

and

$$d'_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})} \simeq d'.$$

Now let $\mathfrak{u}$ be a sequence in $\phi(\mathcal{M})$. If $\mathfrak{u}$ is Cauchy with respect to d_ϕ then there must exist $u \in \overline{\phi(\mathcal{M})}$ so that $\mathfrak{u} \to u$. Since $\mathfrak{u} \to u$ we can see that $\mathfrak{u}$ must also be Cauchy with respect to d'_ϕ. This argument implies that $\mathcal{C}(d_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})}) \subset \mathcal{C}(d'_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})})$. The same argument can be used, in the reverse direction, to show that $\mathcal{C}(d'_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})}) \subset \mathcal{C}(d_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})})$. Hence

$$d_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})} \simeq d'_\phi|_{\phi(\mathcal{M})\times\phi(\mathcal{M})}$$

and since $\simeq$ is an equivalence relation we can conclude that

$$d \simeq d'$$

as required.

$\square$

These results, once combined, can give us the following corollary which is a converse of corollary 4.9.

Corollary 4.11. *Let d be a distance on $\mathcal{M}$ and $\phi : \mathcal{M} \to \mathcal{M}_\phi$ a non-trivial envelopment. If there exists a homeomorphism $f : \mathcal{M}^d \to \overline{\phi(\mathcal{M})}$ so that $f\imath_d = \phi$, then $\phi \in E(d)$ and $d \in D(\mathcal{M})$.*

Proof. Let d_ϕ be a complete distance on $\mathcal{M}_\phi$. Let $d' = d_\phi|_{\phi(\mathcal{M}) \times \phi(\mathcal{M})}$ then, by definition, $\phi \in E(d')$ and so $d' \in D(\mathcal{M})$. From corollary 4.9 there must exist a homeomorphism $h : \mathcal{M}^{d'} \to \overline{\phi(\mathcal{M})}$ so that $h\imath_{d'} = \phi$.

Let $g : \mathcal{M}^d \to \mathcal{M}^{d'}$ be defined by $g = h^{-1}f$. Then g is a homeomorphism and $g\imath_d = h^{-1}f\imath_d = h^{-1}\phi = \imath_{d'}$. Hence, by proposition 4.8, we can see that $d \simeq d'$. Therefore, from definition 4.5, $\phi \in E(d)$ and $d \in D(\mathcal{M})$ as required. □

4.1 The alternative proof of proposition 4.8

Just as we gave an alternative proof of proposition 3.5 using the extension theorem 3.11 for Cauchy continuous functions, we now give an alternative proof of proposition 4.8.

Alternative proof of proposition 4.8. Suppose that $d \simeq d'$. Give $\mathcal{M}$ the Cauchy structure, C_d, induced by d and give $\mathcal{M}^{d'}$ the Cauchy structure, $C_{d'}$, given by d'. That is, $\mathfrak{s} \in C_d$ ($\mathfrak{s} \in C_{d'}$) if and only if $\mathfrak{s}$ is Cauchy with respect to d (d'^*). Note that $\mathfrak{s} \in C_d$ implies $\mathfrak{s} \subset \mathcal{M}$ while $\mathfrak{s} \in C_{d'}$ implies that $\mathfrak{s} \subset \mathcal{M}^{d'}$. Define $f : \mathcal{M} \to \mathcal{M}^{d'}$ by $f(x) = \imath_{d'}(x)$. Essentially we will show that $\imath_{d'}$ is Cauchy continuous with respect to the Cauchy structure induced by d.

Let $\mathfrak{s} \in C_d$. If $\mathfrak{s}$ converges to s in $\mathcal{M}$ then, by the continuity of $\imath_{d'}$, we know that $f(\mathfrak{s}) = \imath_{d'}(\mathfrak{s})$ must converge to $\imath_{d'}(s)$. That is, $f(\mathfrak{s})$ is Cauchy with respect to d'^*. So suppose that $\mathfrak{s} \in C_d$ but does not have a limit point in $\mathcal{M}$. We then know that $\mathfrak{s} \in C(d)$. As $C(d) = C(d')$ we can immediately conclude that $f(\mathfrak{s}) \in C_{d'}$ and hence f is Cauchy continuous. Thus, by theorem 3.11, there exists a unique extension of f, $\hat{f} : \mathcal{M}^d \to \mathcal{M}^{d'}$ so that $\hat{f}\imath_d = \imath_{d'}$. Moreover, by the uniqueness of $\hat{f}$ and, as the argument above is symmetric in d and d', we can see that $\hat{f}$ must be a homeomorphism.

Suppose that we have such an f. Let $\mathfrak{s} \in C(d)$. As f is continuous, $f\imath_d(\mathfrak{s})$ must be Cauchy with respect to d'^* and cannot have a limit point in $\imath_{d'}(\mathcal{M})$. Therefore $\mathfrak{s} \in C(d')$. By symmetry we can conclude that $C(d) = C(d')$ as required. □

5 A correspondence between the equivalence classes

We begin with some definitions.

Definition 5.1. *Let Φ be the set of all envelopments of $\mathcal{M}$.*

Definition 5.2. *Let $[\phi]$ denote the equivalence class of $\phi \in \Phi$ under the equivalence relation $\simeq$.*

Definition 5.3. *Let $[d]$ denote the equivalence class of $d \in D(\mathcal{M})$ under the equivalence relation $\simeq$.*

We will show that there is a one-to-one correspondence between $\frac{D(\mathcal{M})}{\simeq}$ and $\frac{\Phi}{\simeq}$ by constructing two functions which are inverses of each other[9]. First, we give the function from $\frac{D(\mathcal{M})}{\simeq}$ to $\frac{\Phi}{\simeq}$.

Definition 5.4. *For each $d \in D(\mathcal{M})$ choose $\phi \in E(d)$. We shall denote this chosen envelopment by ϕ_d. Define $I : \frac{D(\mathcal{M})}{\simeq} \to \frac{\Phi}{\simeq}$ by letting $I([d]) = [\phi_d]$.*

Lemma 5.5. *The function I is well defined.*

Proof. Let $d, d' \in D(\mathcal{M})$ such that $d \simeq d'$. Since $\phi_d \in E(d)$ and $\phi_{d'} \in E(d')$ there exist two homeomorphisms $h : \mathcal{M}^d \to \overline{\phi_d(\mathcal{M})}$ and $f : \mathcal{M}^{d'} \to \overline{\phi_{d'}(\mathcal{M})}$. Also, as $d \simeq d'$ there exists a homeomorphism $g : \mathcal{M}^d \to \mathcal{M}^{d'}$. Therefore $fgh^{-1} : \overline{\phi_d(\mathcal{M})} \to \overline{\phi_{d'}(\mathcal{M})}$ is a homeomorphism. By construction, $fgh^{-1}\phi_d = \phi_{d'}$, and therefore, by proposition 3.5, $\phi_d \simeq \phi_{d'}$. Thus $I([d]) = I([d'])$ and I must be well defined. $\square$

Now we construct the function from $\frac{\Phi}{\simeq}$ to $\frac{D(\mathcal{M})}{\simeq}$.

Definition 5.6. *Let $\psi \in \Phi$ and choose $d : \mathcal{M}_\psi \times \mathcal{M}_\psi \to \mathbb{R}$ to be a complete distance. Let $d_\psi : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be defined by $d_\psi(x,y) = d(\psi(x), \psi(y))$, for all $x, y \in \mathcal{M}$. Note that, by construction, $\psi \in E(d_\psi)$ and thus $d_\psi \in D(\mathcal{M})$. Define $J : \frac{\Phi}{\simeq} \to \frac{D(\mathcal{M})}{\simeq}$ by $J([\psi]) = [d_\psi]$.*

Lemma 5.7. *The function J is well defined.*

Proof. Let $\psi, \phi \in \Phi$ such that $\psi \simeq \phi$, then there exists a homeomorphism $g : \overline{\psi(\mathcal{M})} \to \overline{\phi(\mathcal{M})}$. Also, since $\psi \in E(d_\psi)$ and $\phi \in E(d_\phi)$ there exist homeomorphisms $h : \mathcal{M}^{d_\phi} \to \overline{\phi(\mathcal{M})}$, $f : \mathcal{M}^{d_\psi} \to \overline{\psi(\mathcal{M})}$. Hence we have the homeomorphism $f^{-1}g^{-1}h : \mathcal{M}^{d_\phi} \to \mathcal{M}^{d_\psi}$. By construction, $f^{-1}g^{-1}h\imath_{d_\phi} = \imath_{d_\psi}$, implying $d_\phi \simeq d_\psi$, by proposition 4.8. Therefore $J([\psi]) = J([\phi])$ so that J is well defined. $\square$

As presented I and J are dependent on a choice of an envelopment and a distance, respectively. It turns out that this choice is immaterial.

Lemma 5.8. *For each $d \in D(\mathcal{M})$ the function I is independent of the choice of ϕ_d.*

Proof. Let $d \in D(\mathcal{M})$, then in order to prove the result we need to show that if $\psi \in E(d)$ then $\psi \simeq \phi_d$.

As $\psi \in E(d)$ we know, from corollary 4.9, that there exists a homeomorphism $f : \mathcal{M}^d \to \overline{\psi(\mathcal{M})}$ so that $f\imath_d = \psi$. Likewise, there must also exist a homeomorphism $g : \mathcal{M}^d \to \overline{\phi_d(\mathcal{M})}$ so that $g\imath_d = \phi_d$. Let $h = fg^{-1}$, so that $h : \overline{\phi_d(\mathcal{M})} \to \overline{\psi(\mathcal{M})}$ is a homeomorphism. Also we know that $h\phi_d = fg^{-1}\phi_d = f\imath_d = \psi$ so that, by proposition 3.5, we can see that $\psi \simeq \phi_d$ as required. $\square$

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[9]In principle this correspondence can be derived using theorem 3.11 by comparing the Cauchy structures induced by an envelopment $\phi : \mathcal{M} \to \mathcal{M}_\phi$ and the Cauchy structure induced by a complete distance on $\mathcal{M}_\phi$. We choose not to follow this route, opting instead to give explicit definitions for the various maps. This does not entail additional complications in the proofs.

Lemma 5.9. *For each $\phi \in \Phi$ the function J is independent of the choice of d_ϕ.*

Proof. Let $\phi : \mathcal{M} \to \mathcal{M}_\phi$ be an element of Φ then, in order to prove our result, we need to show that for any two complete distances $d : \mathcal{M}_\phi \times \mathcal{M}_\phi \to \mathbb{R}$ and $d' : \mathcal{M}_\phi \times \mathcal{M}_\phi \to \mathbb{R}$ the induced distances $d_\phi : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ and $d'_\phi : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$, given by $d_\phi(x, y) = d(\phi(x), \phi(y))$ and $d'_\phi(x, y) = d'(\phi(x), \phi(y))$ (for all $x, y \in \mathcal{M}$) are equivalent. That is, we need to show that $\mathcal{C}(d_\phi) = \mathcal{C}(d'_\phi)$.

Let $\mathfrak{s} \in \mathcal{C}(d_\phi)$ then $\mathfrak{s}$ is Cauchy with respect to d_ϕ and since d is complete, by construction, there must exist $p \in \partial(\phi(\mathcal{M}))$ so that $\phi(\mathfrak{s}) \to p$ uniquely. This implies, however, that $\phi(\mathfrak{s})$ will be Cauchy with respect to any distance on $\mathcal{M}_\phi$ and therefore $\phi(\mathfrak{s})$ is Cauchy with respect to d'. Hence, by construction, $\mathfrak{s}$ is Cauchy with respect to d'_ϕ. Therefore $\mathcal{C}(d_\phi) \subset \mathcal{C}(d'_\phi)$.

Similarly we can see that $\mathcal{C}(d'_\phi) \subset \mathcal{C}(d_\phi)$ and thus $d_\phi \simeq d'_\phi$ as required. $\qquad\square$

Now we present our main results.

Lemma 5.10. *Let $d \in D(\mathcal{M})$, then $JI([d]) = [d]$.*

Proof. Let $I([d]) = [\phi_d]$, then $\phi_d \in E(d)$ so there exists a homeomorphism $f : \mathcal{M}^d \to \overline{\phi_d(\mathcal{M})}$. Since $J([\phi_d]) = [d_{\phi_d}]$ we know that there exists a homeomorphism $g : \mathcal{M}^{d_{\phi_d}} \to \overline{\phi_d(\mathcal{M})}$. Thus $f^{-1}g : \mathcal{M}^{d_{\phi_d}} \to \mathcal{M}^d$ is a homeomorphism. By consulting the definitions we can see that $f^{-1}g\imath_{d_{\phi_d}} = \imath_d$ and therefore $d_{\phi_d} \simeq d$. Hence $JI([d]) = [d_{\phi_d}] = [d]$. $\qquad\square$

Lemma 5.11. *Let $\psi \in \Phi$, then $IJ([\psi]) = [\psi]$.*

Proof. Let $J([\psi]) = [d_\psi]$ then there exists a homeomorphism $f : \mathcal{M}^{d_\psi} \to \overline{\psi(\mathcal{M})}$. Let $I([d_\psi]) = [\phi_{d_\psi}]$ then there exists a homeomorphism $g : \mathcal{M}^{d_\psi} \to \overline{\phi_{d_\psi}(\mathcal{M})}$. Since $fg^{-1} : \overline{\phi_{d_\psi}(\mathcal{M})} \to \overline{\psi(\mathcal{M})}$ is a homeomorphism and as $fg^{-1}\phi_{d_\psi} = \psi$ we conclude that $\psi \simeq \phi_{d_\psi}$. Therefore $IJ([\psi]) = [\phi_{d_\psi}] = [\psi]$. $\qquad\square$

Theorem 5.12. *The function $I : \frac{D(\mathcal{M})}{\simeq} \to \frac{\Phi}{\simeq}$ is a bijective function with inverse J. That is, the sets $\frac{D(\mathcal{M})}{\simeq}$ and $\frac{\Phi}{\simeq}$ are in one-to-one correspondence with each other.*

Proof. This follows from lemmas 5.10 and 5.11. $\qquad\square$

This theorem shows us that any information that can be extracted from $\frac{\Phi}{\simeq}$ (e.g., the a-boundary) can also be extracted from $\frac{D(\mathcal{M})}{\simeq}$.

6 Demonstration of correspondence

To illustrate how this correspondence can be used we show how the 'covering' and 'in contact' relations between boundary points of the two maximal extensions of the Misner space-time, for $t > 0$, can be constructed using envelopments or their corresponding distances.

6.1 The Misner space-time

The upper-half Misner space-time is the space-time with manifold $\mathcal{M} = \mathbb{R}^+ \times S^1$ with metric, in the coordinates t and ψ, $0 < t < \infty$, $0 \leq \psi < 2\pi$, given by

$$g = \begin{pmatrix} \frac{-1}{t} & 0 \\ 0 & t \end{pmatrix}.$$

It is well known [9] that there exist two maximal extensions of $\mathcal{M}$. Let $\mathcal{M}_1 = \mathbb{R} \times S^1$ with coordinates t, ψ_1, $t \in \mathbb{R}$, $0 \leq \psi_1 < 2\pi$ and metric

$$g_1 = \begin{pmatrix} 0 & 1 \\ 1 & t \end{pmatrix}.$$

Let $\mathcal{M}_2 = \mathbb{R} \times S^1$ with coordinates t, ψ_2, $t \in \mathbb{R}$, $0 \leq \psi_2 < 2\pi$ and metric

$$g_2 = \begin{pmatrix} 0 & -1 \\ -1 & t \end{pmatrix}.$$

There are two envelopments $\phi_1 : \mathcal{M} \to \mathcal{M}_1$ and $\phi_2 : \mathcal{M} \to \mathcal{M}_2$ given by

$$\phi_1(t, \psi) = (t, \psi - \log t \mod 2\pi)$$
$$\phi_2(t, \psi) = (t, \psi + \log t \mod 2\pi).$$

In both cases $\phi_i(\mathcal{M})$ is isometric to $\mathcal{M}$, $i = 1, 2$. These manifolds and the maps between them provide a wealth of counter-examples for various conjectures in General Relativity.

6.2 A useful sequence

For the purposes of this section it is important to construct a sequence in $\mathbb{R}^+$ having certain properties. Let $r \in \mathbb{Q} \cap (0, 2\pi)$. Let $t_n = nr$, for all $n \in \mathbb{N}$. Since r is rational we know that the set $\{t_n \mod 2\pi\}$ is dense in $[0, 2\pi)$. Let $s_n = \exp(-t_n)$.

6.2.1 Construction of the 'in contact' and 'covering' relation using envelopments

Choose $(0, \psi_1) \in \partial(\phi_1(\mathcal{M}))$ and $(0, \psi_2) \in \partial(\phi_2(\mathcal{M}))$. We will show that these two arbitrary points on the boundaries are in contact and that neither covers the other. Take the sequence $\mathfrak{s} = \{(\sqrt{s_n}, \psi_1 + \log\sqrt{s_n} \mod 2\pi)\}$ in $\mathcal{M}$ so that $\phi_1(\mathfrak{s}) = \{(\sqrt{s_n}, \psi_1)\}$ clearly converges to $(0, \psi_1)$. We can calculate that $\phi_2(\mathfrak{s}) = \{(\sqrt{s_n}, \psi_1 + 2\log\sqrt{s_n} \mod 2\pi)\} = \{(\sqrt{s_n}, \psi_1 - t_n \mod 2\pi)\}$. Since $\{t_n \mod 2\pi\}$ is dense in $[0, 2\pi)$ there exists a subsequence $\{u_n\} \subset \{t_n\}$ so that $\{u_n \mod 2\pi\}$ converges to $\psi_2 - \varphi \mod 2\pi$, where $\varphi \in [0, 2\pi)$ is arbitrary. Let $q_n = \exp(-u_n)$ and $\mathfrak{q} = \{(\sqrt{q_n}, \psi_1 + \log\sqrt{q_n} \mod 2\pi)\}$. It is clear that $\mathfrak{q}$ is a subsequence of $\mathfrak{s}$ so that $\phi_1(\mathfrak{q})$ must converge to $(0, \psi_1)$. By construction we also have the following:

$$\lim_{n\to\infty} \phi_2(\mathfrak{q}) = \lim_{n\to\infty} \{(\sqrt{q_n}, \psi_1 + 2\log\sqrt{q_n} \mod 2\pi)\}$$
$$= \lim_{n\to\infty} \{(\sqrt{q_n}, \psi_1 - u_n \mod 2\pi)\}$$
$$= (0, \psi_1 - \psi_2 + \varphi \mod 2\pi).$$

Since $\varphi \in [0, 2\pi)$ is arbitrary, the calculation above shows that $\phi_2(\mathfrak{s})$ has every point of $\partial(\phi_2(\mathcal{M}))$ as a limit point. In particular $\phi_2(\mathfrak{q})$ converges to $(0, \psi_2)$ for $\varphi = 2\psi_2 - \psi_1$ mod 2π. By symmetry this is enough to prove that $(0, \psi_1) \in \partial(\phi_1(\mathcal{M}))$ and $(0, \psi_2) \in \partial(\phi_2(\mathcal{M}))$ are in contact but that neither covers the other.

6.2.2 Construction of the 'in contact' and 'covering' relation using distances

Members of the equivalence classes of distances on $\mathcal{M}$ induced by ϕ_1 and ϕ_2 can be calculated as follows. First we take two complete distances, d_1 and d_2, on $\mathcal{M}_1$ and $\mathcal{M}_2$. These are, respectively:

$$d_1\big((t, \psi_1), (t', \psi_1')\big) = \sqrt{(t - t')^2 + (\psi_1 - \psi_1')^2}$$
$$d_2\big((t, \psi_2), (t', \psi_2')\big) = \sqrt{(t - t')^2 + (\psi_2 - \psi_2')^2}.$$

Next we pull these back to $\mathcal{M}$. In an abuse of notation we will also denote the pull backs by d_1 and d_2. The result is

$$d_1\big((t, \psi), (t', \psi')\big) = \sqrt{(t - t')^2 + \big((\psi - \log t \mod 2\pi) - (\psi' - \log t' \mod 2\pi)\big)^2}$$
$$d_2\big((t, \psi), (t', \psi')\big) = \sqrt{(t - t')^2 + \big((\psi + \log t \mod 2\pi) - (\psi' + \log t' \mod 2\pi)\big)^2}.$$

These are representatives of the equivalence classes of distances which correspond to the equivalence classes of ϕ_1 and ϕ_2.

Let $\psi_1' \in [0, 2\pi)$. We can check that the sequence $\mathfrak{s}_1 = \{(\sqrt{s_n}, \psi_1' + \log \sqrt{s_n} \mod 2\pi)\}$ in $\mathcal{M}$ is Cauchy with respect to d_1:

$$d_1\big((\sqrt{s_n}, \psi_1' + \log \sqrt{s_n} \mod 2\pi), (\sqrt{s_m}, \psi_1' + \log \sqrt{s_m} \mod 2\pi)\big)$$
$$= \sqrt{(\sqrt{s_n} - \sqrt{s_m})^2}$$
$$= |\sqrt{s_n} - \sqrt{s_m}|.$$

As $\{s_n\}$ converges to 0, we can take n and m large enough to make the distance above arbitrarily small. Therefore $\mathfrak{s}_1$ is Cauchy with respect to d_1. From corollary 4.9 and definition 5.6 we can see that $[\{(\sqrt{s_n}, \psi_1' + \log \sqrt{s_n} \mod 2\pi)\}]_{d_1}$ corresponds to the boundary point $(0, \psi_1')$ in $\partial(\phi_1(\mathcal{M}))$. Repeating this argument for ϕ_2 we see that the sequence $\mathfrak{s}_2 = \{(\sqrt{s_n}, \psi_2' - \log \sqrt{s_n} \mod 2\pi)\}$ is Cauchy with respect to d_2 and corresponds to the boundary point $(0, \psi_2')$ in $\partial(\phi_2(\mathcal{M}))$.

We may now calculate the distance between these sequences with respect to d_1:

$$d_1\big((\sqrt{s_n}, \psi_1' + \log \sqrt{s_n} \mod 2\pi), (\sqrt{s_m}, \psi_2' - \log \sqrt{s_m} \mod 2\pi)\big)$$
$$= \sqrt{(\sqrt{s_n} - \sqrt{s_m})^2 + (\psi_1' - \psi_2' - t_m \mod 2\pi)^2}.$$

Since $\{t_m \mod 2\pi\}$ is dense in $[0, 2\pi)$ we can immediately see that for all $\epsilon < 0$ there exists $n_0, m_0 \in \mathbb{N}$ so that

$$d_1\big((\sqrt{s_{n_0}}, \psi_1' + \log\sqrt{s_{n_0}} \mod 2\pi), (\sqrt{s_{m_0}}, \psi_2' - \log\sqrt{s_{m_0}} \mod 2\pi)\big) < \epsilon$$

but that there does not exist $N \in \mathbb{N}$ so that for all $n, m > N$ this is true.

If the points $[\mathfrak{s}_1]_{d_1}$ and $[\mathfrak{s}_2]_{d_2}$ covered each other then the distance, above, would have to limit to zero as $n, m \to \infty$. We have just demonstrated that the distance is not zero in the limit and therefore neither point covers the other. Moreover there exists $\mathfrak{q} \subset \mathfrak{s}_2$ so that $[\{(\sqrt{s_n}, \psi_1' + \log\sqrt{s_n} \mod 2\pi)\}]_{d_1} = [\mathfrak{q}]_{d_1}$ and therefore the points $[\mathfrak{s}_1]_{d_1}$ and $[\mathfrak{s}_2]_{d_2}$ are in contact. Thus, under the correspondence, we know that $(0, \psi_1') \in \partial(\phi_1(\mathcal{M}))$ and $(0, \psi_2') \in \partial(\phi_2(\mathcal{M}))$ are in contact but neither covers the other.

6.3 Discussion

The construction of the relations via distances and envelopments are of a similar complexity but use very different techniques. It is this difference that this paper advocates. One now has a choice of techniques for working with the Abstract Boundary. We note that until an envelopment independent definition of $D(\mathcal{M})$ is given there is still some level of dependence on envelopments. This came through above via the use of pull backs to define our distances. This problem is therefore of a pressing nature. Unfortunately this is a very difficult problem since it requires a characterization of the distances on the manifold which correspond to envelopments.

That every boundary point in $\partial(\phi_1(\mathcal{M}))$ is 'in contact' with every boundary point in $\partial(\phi_2(\mathcal{M}))$ but no two points cover each other expresses the fact that the boundary $\partial(\phi_1(\mathcal{M}))$ is 'smeared' over the boundary $\partial(\phi_2(\mathcal{M}))$ and vice versa. The sets σ_{ϕ_1} and σ_{ϕ_2} are therefore each partial cross sections which contain the same boundary information expressed in very different ways. The boundary information associated with a point of $\partial(\phi_1(\mathcal{M}))$ is spread over every boundary point in $\partial(\phi_2(\mathcal{M}))$. This, very odd, behaviour gives an example of how the Abstract Boundary copes with multiple maximal envelopments.

7 Conclusions

We have defined equivalence relations on the set of all envelopments of a manifold and a subset of the set of all distances on a manifold. The resulting sets of equivalence classes were then shown to be in one-to-one correspondence with each other; hence they are 'the same.' Since the Abstract Boundary can be constructed from $\frac{\Phi}{\simeq}$, we can conclude that it is possible to construct the Abstract Boundary using $\frac{D(\mathcal{M})}{\simeq}$ instead. In a following paper we will show how this can be done. Therefore, instead of thinking about boundary points of a particular envelopment of the manifold, we can now think about collections of Cauchy sequences with respect to some distance on the manifold.

The Abstract Boundary has already proven to be a very useful, intuitive construction (see [2]). By showing how envelopments can be replaced by distances, the construction can now be applied in new ways. Note, however, that to define the set $D(\mathcal{M})$ we had to refer to envelopments, via the sets $E(d)$. So, while we have presented an alternative way to view the Abstract Boundary, in order to fully dissociate the two approaches (distances vs. envelopments) we still need to find a definition for $D(\mathcal{M})$ that does not in any way rely on envelopments. Current research is pursuing this goal.

In summary, we have shown that the structure of the edge of a space-time can be deduced from knowledge of a distance defined purely on the space-time itself. Given the successes of the Abstract Boundary, this thereby provides new tools when working with the edge of a space-time.

By giving the relationship between $\frac{\Phi}{\simeq}$ and $\frac{D(\mathcal{M})}{\simeq}$ we have demonstrated how a fundamental building block of the Abstract Boundary can be replaced when required.

8 Acknowledgements

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